

# theories of integration the integrals of riemann lebesgue henstock kurzweil and mcshane second edition series in real analysis

The theories of integration have long been a fundamental aspect of real analysis, providing a framework for understanding the accumulation of quantities. Among the most notable theories are the Riemann, Lebesgue, Henstock, Kurzweil, and McShane integrals. Each of these integration methods offers unique perspectives and applications, enhancing our comprehension of functions and their integrals. In this article, we will delve into each of these theories, comparing their methodologies and implications.

## 1. Riemann Integral

The Riemann integral is the cornerstone of integral calculus, introduced by Bernhard Riemann in the 19th century. It defines the integral of a function as the limit of Riemann sums.

### 1.1 Definition and Construction

To define the Riemann integral, consider a bounded function  $f: [a, b] \rightarrow \mathbb{R}$ . The Riemann sum is constructed by partitioning the interval  $[a, b]$  into  $n$  subintervals, each of width  $\Delta x_i = x_i - x_{i-1}$ . The Riemann sum is given by:

$$S_n = \sum_{i=1}^n f(x_i^*) \Delta x_i$$

where  $x_i^*$  is a sample point in the  $i$ -th subinterval. The Riemann integral is defined as:

$$\int_a^b f(x) \, dx = \lim_{n \rightarrow \infty} S_n$$

if this limit exists.

## 1.2 Properties

- Linearity: If  $f$  and  $g$  are integrable, then  $\int (af + bg) \, dx = a \int f \, dx + b \int g \, dx$  for constants  $a$  and  $b$ .
- Additivity: If  $f$  is integrable on  $[a, b]$  and  $[b, c]$ , then  $\int_a^c f \, dx = \int_a^b f \, dx + \int_b^c f \, dx$ .

## 1.3 Limitations

The Riemann integral has limitations, particularly when dealing with functions that are not continuous or have too many discontinuities. For example, the characteristic function of the rationals is not Riemann integrable due to its discontinuities.

## 2. Lebesgue Integral

The Lebesgue integral, developed by Henri Léon Lebesgue in the early 20th century, overcomes many of the limitations of the Riemann integral by focusing on the measure of sets rather than intervals.

## 2.1 Definition and Construction

The Lebesgue integral is defined using measurable functions and measures. For a non-negative measurable function  $f$ , the integral is defined as:

$$\int f \, d\mu = \sup \left\{ \sum_{k=1}^{\infty} a_k \mu(E_k) : f \leq \sum_{k=1}^{\infty} a_k \chi_{E_k} \right\}$$

where  $E_k$  are measurable sets and  $\chi_{E_k}$  is the characteristic function.

## 2.2 Properties

- Monotonicity: If  $f \leq g$  almost everywhere, then  $\int f \, d\mu \leq \int g \, d\mu$ .
- Dominated Convergence Theorem: If  $f_n \rightarrow f$  almost everywhere and  $|f_n| \leq g$  for some integrable function  $g$ , then  $\int f_n \, d\mu \rightarrow \int f \, d\mu$ .

## 2.3 Advantages

The Lebesgue integral can integrate a broader class of functions, including those with infinite discontinuities, and is essential in probability theory and functional analysis.

## 3. Henstock Integral

The Henstock integral, also known as the generalized Riemann integral, extends the Riemann integral by allowing variable partitions.

### 3.1 Definition and Construction

The Henstock integral is defined using a gauge function that permits different widths of partitions based on the behavior of the function. A function  $f$  is Henstock integrable if for every  $\epsilon > 0$ , there exists a gauge  $\delta$  such that:

$$\left| \sum_{i=1}^n f(\xi_i) (x_i - x_{i-1}) \right| < \epsilon$$

for any tagged partition satisfying the gauge.

### 3.2 Properties

- Includes Riemann Integral: Every Riemann integrable function is Henstock integrable.
- Integration by Parts: The Henstock integral allows for a more general form of integration by parts.

## 4. Kurzweil Integral

The Kurzweil integral is closely related to the Henstock integral and was developed by the mathematician Helmut Kurzweil.

### 4.1 Definition and Construction

Similar to the Henstock integral, the Kurzweil integral uses a partitioning strategy but emphasizes the use of a suitable gauge. The integral is defined through the concept of a "generalized Riemann sum".

## 4.2 Properties

- Generalization: It generalizes the concept of the Riemann integral and can handle more complex functions.
- Connection to Differential Equations: The Kurzweil integral is particularly useful in solving certain types of differential equations.

## 5. McShane Integral

The McShane integral is another extension of the Riemann integral that provides a useful framework for integration.

### 5.1 Definition and Construction

The McShane integral is defined via a process similar to Riemann sums but incorporates the concept of upper and lower sums. It allows for the integration of functions that may not be Riemann integrable.

### 5.2 Properties

- Flexibility: The McShane integral can integrate functions with a wider range of discontinuities compared to the Riemann integral.
- Relation to Lebesgue Integral: The McShane integral is related to the Lebesgue integral, providing a bridge between the two theories.

## 6. Conclusion

The theories of integration—Riemann, Lebesgue, Henstock, Kurzweil, and McShane—each contribute unique insights into the nature of integrals and the functions they encompass. While the Riemann integral serves as the foundational concept, the Lebesgue integral expands the scope of integrability in a measure-theoretic context. The Henstock and Kurzweil integrals provide further generalizations, allowing for flexibility in handling functions with discontinuities. Finally, the McShane integral adds another layer of depth to the understanding of integration.

Understanding these theories of integration not only enriches the study of real analysis but also lays the groundwork for advanced topics in mathematics, including functional analysis and probability theory. As we continue to explore these concepts, the interplay between different integration theories reveals the complexity and beauty of mathematical analysis.

## Frequently Asked Questions

### What is the Riemann integral and how does it differ from the Lebesgue integral?

The Riemann integral focuses on summing the areas of rectangles under a curve and is limited to functions that are bounded and defined on closed intervals. In contrast, the Lebesgue integral extends the concept by measuring the 'size' of the set where the function takes certain values, allowing for integration of more complex functions, including those that are unbounded or defined on more abstract spaces.

### What is the significance of the Lebesgue Dominated Convergence Theorem?

The Lebesgue Dominated Convergence Theorem provides conditions under which the limit of an

integral can be interchanged with the integral of a limit. This theorem is crucial for proving the convergence of integrals and is widely used in analysis, especially in probability theory and functional analysis.

## **How do the Henstock–Kurzweil and McShane integrals differ from the Riemann and Lebesgue integrals?**

The Henstock-Kurzweil and McShane integrals generalize the Riemann integral by allowing for variable-sized partitions and the integration of a broader class of functions. Unlike the Lebesgue integral, which relies on measure theory, these integrals can handle more pathological cases, such as certain oscillatory functions, while still retaining a geometric intuition similar to Riemann integration.

## **What are the main advantages of using the Henstock–Kurzweil integral over the Lebesgue integral?**

The Henstock-Kurzweil integral allows for the integration of functions that are not Lebesgue integrable, such as those with unbounded oscillation. It also retains some properties of the Riemann integral, making it easier to understand and compute in certain cases, while still being more powerful than traditional Riemann integration.

## **What is the role of the 'gauge' in the Henstock–Kurzweil integral?**

In the Henstock-Kurzweil integral, a 'gauge' is a function that assigns a positive number to each point in the interval, dictating how finely the interval should be partitioned at that point. This allows for a more flexible approach to integration, accommodating varying degrees of precision based on the behavior of the function being integrated.

## **Can you explain the concept of 'absolute continuity' in relation to integration?**

Absolute continuity is a stronger form of continuity for functions that implies not only that the function can be represented as an integral of its derivative, but also that small changes in the input lead to

small changes in the output uniformly. This concept is important in the context of both Riemann and Lebesgue integrals, as it ensures the function behaves well under integration.

## What are some applications of the McShane integral?

The McShane integral is particularly useful in fields requiring integration of functions that may be discontinuous or have complex behavior. Its applications include areas of analysis, such as solving differential equations, and in probability theory for dealing with random variables that exhibit discontinuities.

## How do the concepts of measure theory and integration interrelate in the context of Lebesgue and Henstock–Kurzweil integrals?

Measure theory provides the foundational framework for the Lebesgue integral, focusing on the size of sets and the properties of measurable functions. In contrast, the Henstock-Kurzweil integral does not rely on measure theory, offering a more intuitive approach to integration while still accommodating a broad class of functions, revealing the deep connections between these different approaches to integration.

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